Information-Theoretic Measure of Genuine Multi-Qubit Entanglement

Jian-Ming Cai[1], Zheng-Wei Zhou[1]* and Xing-Xiang Zhou[2], and Guang-Can Guo[1]
[1] *Laboratory of Quantum Information, University of Science and Technology of China, Hefei, Anhui 230026, P. R. China*
[2] *Physics Department, Pennsylvania State University, University Park, Pennsylvania 16802, USA*

We consider pure quantum states of N qubits and study the genuine $N-$qubit entanglement that is shared among all the N qubits. We introduce an information-theoretic measure of *genuine N-qubit entanglement* based on bipartite partitions. When N is an even number, this measure is presented in a simple formula, which depends only on the purities of the partially reduced density matrices. It can be easily computed theoretically and measured experimentally. When N is an odd number, the measure can also be obtained in principle.

PACS numbers: 03.67.-a, 03.65.Ud, 73.43.Nq, 89.70.+c

The nature of quantum entanglement is a fascinating topic in quantum mechanics since the famous Einstein-Podolsky-Rosen paper [1] in 1935. Recently, much interest has been focused on entanglement in quantum systems containing a large number of particles. On one hand, multipartite entanglement is valuable physical resource in large-scale quantum information processing [2, 3]. On the other hand, multipartite entanglement seems to play an important role in condensed matter physics [4], such as quantum phase transitions (QPT) [5, 6] and high temperature superconductivity [7]. Therefore, how to characterize and quantify multipartite entanglement remains one of the central issues in quantum information theory.

In the present literature, there exist very few measures of multipartite entanglement with a clear physical meaning [8, 9, 10, 11, 12, 13, 14, 15]. Because of this, most research in quantum entanglement and QPT focused on bipartite entanglement, for which there have been several well defined measures [16, 17, 18, 19, 20, 21, 22]. However, bipartite entanglement can not characterize the global quantum correlations among all parties in a multiparticle system. Since the correlation length diverges at the critical points, multipartite entanglement plays an essential role in QPT. Though localizable entanglement (LE) [23] can be used to describe long-range quantum correlations, its determination is a formidable task for generic pure states. Therefore, computable measure of multipartite entanglement with clear physical meanings are highly desired [24, 25].

In this paper, we define a new measure of genuine $N-$qubit entanglement based on different bipartite partitions of the qubits and existing measures for mutual information. The central idea is that, through bipartite partitions, we can get information about the genuine multi-qubit entanglement. Our measure is a polynomial SLOCC (stochastic local operations and classical communication) invariant [26] and is unchanged under permutations of qubits. When N is an even number, we derive a simple formula for this measure, which is determined by the purity of partially reduced density matrices only. It can be computed through experimentally observable quantities [10, 27]. Therefore, it is easy to obtain not only theoretically but also experimentally. For $N=4$, we show that this measure satisfies all the necessary conditions required for a natural entanglement measure [20] exactly. When N is an odd number, the measure for genuine N-qubit entanglement is defined on the basis of the measure for even number of qubits. This measure will definitely extend the research in the field of multipartite entanglement and condensed matter systems.

Bipartite partition and genuine multi-qubit entanglement For multi-qubit pure states, there exist local information, and nonlocal information which is related to quantum correlations [28]. In a closed two- and three-qubit system, i.e. the system state is pure and its evolution is unitary, we have shown that [29] entanglement is relevant to some kind of nonlocal information which contributes to the conserved total information. Consider a pure state $|\psi\rangle$ of N qubits, labelled as $1, 2, \cdots, N$, generally we can write $|\psi\rangle = |\psi_1\rangle\otimes\cdots\otimes|\psi_M\rangle$ [25], where $|\psi_m\rangle$ are non-product pure states, $m = 1, 2, \cdots, M$, and the qubits of different $|\psi_m\rangle$ have no intersection. If $M = 1$, $|\psi\rangle$ itself is a non-product pure state, otherwise $|\psi\rangle$ is a product pure state. A bipartite partition $\mathcal{P}$ will divide the qubits of $|\psi_m\rangle$ into two subsets $\mathcal{A}_m$ and $\mathcal{B}_m$, then all the N qubits are divided into $\mathcal{A} = \cup_{m=1}^{M}\mathcal{A}_m$ and $\mathcal{B} = \cup_{m=1}^{M}\mathcal{B}_m$, e.g. see Fig.1(a). In this paper, we use the linear entropy [30], then the mutual information between $\mathcal{A}_m$ and $\mathcal{B}_m$ is $I_{\mathcal{A}_m\mathcal{B}_m} = S_{\mathcal{A}_m} + S_{\mathcal{B}_m} - S_{\mathcal{A}_m\mathcal{B}_m}$, where $S_Y = 1-Tr\rho_Y^2$, $Y = \mathcal{A}_m, \mathcal{B}_m, \mathcal{A}_m\mathcal{B}_m$. Since $|\psi_m\rangle$ is pure, we can write $I_{\mathcal{A}_m\mathcal{B}_m} = 2(1 - Tr\rho^2_{\mathcal{A}_m}) = 2(1 - Tr\rho^2_{\mathcal{B}_m})$, where $\rho_{\mathcal{A}_m}$ and $\rho_{\mathcal{B}_m}$ are the reduced density matrices. The bipartite nonlocal information between $\mathcal{A}$ and $\mathcal{B}$, denoted as $S_{\mathcal{A}|\mathcal{B}}$, is defined as the sum of mutual information between $\mathcal{A}_m$ and $\mathcal{B}_m$,

$$S_{\mathcal{A}|\mathcal{B}} := \sum_{m=1}^{M} I_{\mathcal{A}_m\mathcal{B}_m} \tag{1}$$

If $\mathcal{A}_m$ or $\mathcal{B}_m$ is empty we set $I_{\mathcal{A}_m\mathcal{B}_m} = 0$.

The information diagram for two- and three-qubit pure

*Electronic address: zwzhou@ustc.edu.cn

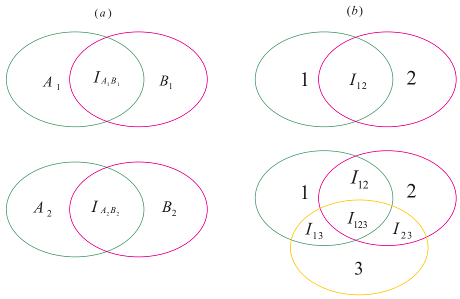

FIG. 1: Information diagram for (a) product pure states $|\psi\rangle_{A_1B_1} \otimes |\psi\rangle_{A_2B_2}$, the bipartite nonlocal information (*overlap*) between $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ is $S_{A|B} = I_{A_1B_1} + I_{A_2B_2}$; (b) two-qubit (top) and three-qubit (bottom) pure states.

states is depicted in Fig.1(b). Each qubit is represented by one circle. The overlap of k different circles, indexed as $i_1, i_2, \cdots, i_k$, represents the nonlocal information $I_{i_1 i_2 \cdots i_k}$ that is *shared among all these k qubits*. If we adopt some appropriate measure of information, i.e. linear entropy [29, 30], the nonlocal information is directly relevant to genuine $k-$qubit entanglement. For example, for two-qubit pure states, $I_{12} = \tau_{12}$, where τ_{12} is the square of concurrence. For three-qubit pure states, the bipartite nonlocal information between 1 and 23 can be written as $S_{1|23} = I_{12} + I_{13} + I_{123}$ with $I_{12} = \tau_{12}$, $I_{13} = \tau_{13}$ and $I_{123} = \tau_{123}$, where 3-tangle τ_{123} has been shown to be a well-defined measure of genuine three-qubit entanglement [31]. For pure states of $N > 3$ qubits, we generalize the above viewpoint of multi-qubit entanglement, assisted by the information diagram, to quantify genuine N-qubit entanglement by the nonlocal information shared by all the N qubits.

Observation: For a pure state of N qubits and a partition $\mathcal{A}|\mathcal{B}$, the bipartite nonlocal information between $\mathcal{A}$ and $\mathcal{B}$ is contributed by different levels of nonlocal information $S_{\mathcal{A}|\mathcal{B}} = \sum_{k=2}^{N} I_{i_1 i_2 \cdots i_k}$, where $i_1, i_2, \cdots, i_k$ are not in the same set $\mathcal{A}$ or $\mathcal{B}$, and $I_{i_1 i_2 \cdots i_k}$ is some appropriate measure of nonlocal information.

Based on the above observation, we propose an information-theoretic measure of genuine $N-$qubit entanglement through bipartite partitions. We utilize this measure to explore the genuine multi-qubit entanglement in spin systems. Our results also suggest that the above observation is reasonable.

Genuine four-qubit entanglement For a pure state of N qubits, where $N \in$ *even*, there are two different classes of bipartite partitions $\mathcal{P}_{\mathcal{I}}$ and $\mathcal{P}_{\mathcal{II}}$. For any partition $\mathcal{P} = \mathcal{A}|\mathcal{B}$, we denote the number of qubits contained in $\mathcal{A}$ and $\mathcal{B}$ as $|\mathcal{A}|$, $|\mathcal{B}|$. If $|\mathcal{A}|, |\mathcal{B}| \in$ *odd*, $\mathcal{P} \in \mathcal{P}_{\mathcal{I}}$, and if $|\mathcal{A}|, |\mathcal{B}| \in$ *even*, $\mathcal{P} \in \mathcal{P}_{\mathcal{II}}$. When $N = 4$, $\mathcal{P}_{\mathcal{I}} = \{1|234, 2|134, 3|124, 4|123\}$ and $\mathcal{P}_{\mathcal{II}} = \{12|34, 13|24, 14|23\}$. For partition $1|234$, the bipartite nonlocal information is denoted as $S_{1|234} = I_{12} + I_{13} + I_{14} + I_{123} + I_{124} + I_{134} + I_{1234}$. Similarly, we can get the bipartite nonlocal information for the other seven partitions, denoted as $S_{2|134}$, $S_{3|124}$, $S_{4|123}$, $S_{12|34}$, $S_{13|24}$ and $S_{14|23}$. It can be seen that $S_{\mathcal{I}} = \sum_{\mathcal{P}\in\mathcal{P}_{\mathcal{I}}} S_{\mathcal{P}} = 2I_{12} + 2I_{13} + 2I_{14} + 2I_{23} + 2I_{24} + 2I_{34} + 3I_{123} + 3I_{124} + 3I_{134} + 3I_{234} + 4I_{1234}$. In the same way, we can get $S_{\mathcal{II}} = \sum_{\mathcal{P}\in\mathcal{P}_{\mathcal{II}}} S_{\mathcal{P}} = S_{\mathcal{I}} - I_{1234}$. Therefore, genuine four-qubit entanglement $\mathcal{E}_{1234}$ can be naturally measured by the nonlocal information I_{1234}, i.e. the difference between $S_{\mathcal{I}}$ and $S_{\mathcal{II}}$

$$\mathcal{E}_{1234} = S_{\mathcal{I}} - S_{\mathcal{II}} \tag{2}$$

The above definition of the measure for genuine four-qubit entanglement must satisfy the following conditions in order to be a natural entanglement measure for pure states [20]. (1) $\mathcal{E}_{1234}$ is invariant under local unitary operations. (2) $\mathcal{E}_{1234} \geq 0$ for all pure states. (3) $\mathcal{E}_{1234}$ is an entanglement monotone, *i.e.* , $\mathcal{E}_{1234}$ does not increase on average under local quantum operations assisted with classical communication (LOCC).

It is obvious that the mutual information $I_{\mathcal{A}_m\mathcal{B}_m}$ is determined by the eigenvalues of the partially reduced density matrices. Therefore, $\mathcal{E}_{1234}$ is invariant under local unitary operations. Moreover, according to the above definitions in Eqs.(1-2), it is easy to verify that $\mathcal{E}_{1234} = 0$ for product pure states.

In order to prove that $\mathcal{E}_{1234}$ satisfies the other two conditions, we first investigate how $\mathcal{E}_{1234}$ will change under determinant 1 SLOCC operations. Determinant 1 SLOCC operations [25, 32] are local operations which transform a pure state $|\psi\rangle$ to $|\psi'\rangle = A_1 \otimes \cdots \otimes A_n|\psi\rangle/Q$ with $Q^2 = Tr(A_1 \otimes \cdots \otimes A_n|\psi\rangle\langle\psi|A_1^\dagger \otimes \cdots \otimes A_n^\dagger)$, where $A_i \in SL(2, C)$ is an operator on the ith qubit. Without loss of generality, we assume that $|\psi\rangle$ is a non-product pure state, and the determinant 1 SLOCC operation is only performed on the $1st$ qubit. The determinant 1 SLOCC operation A_1 can be written as $U \cdot diag[d, 1/d] \cdot V$, where U, $V \in SU(2)$ and d is a positive real number. Since the mutual information is invariant under local unitary operations, we do not have to consider the unitary operations U and V. Let us write the four-qubit state $|\psi\rangle = \sqrt{p_0}|0\rangle|\varphi_0\rangle + \sqrt{p_1}|1\rangle|\varphi_1\rangle$, where $p_0 + p_1 = 1$, $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are pure states of qubits $2, 3, 4$. After the operation A_1, $|\psi'\rangle = \frac{A_1}{Q}|\psi\rangle = \frac{\sqrt{p_0}d}{Q}|0\rangle|\varphi_0\rangle + \frac{\sqrt{p_1}}{Qd}|1\rangle|\varphi_1\rangle$, where $Q = (p_0 d^2 + p_1/d^2)^{1/2}$. Since $\rho'_{234} = (p_0 d^2|\varphi_0\rangle\langle\varphi_0| + p_1/d^2|\varphi_1\rangle\langle\varphi_1|)/Q^2$, $Tr(\rho'_1)^2 = Tr(\rho'_{234})^2 = (p_0^2 d^4 + p_1^2/d^4 + f_1)/Q^4$, where $f_1 = 2p_0p_1Tr(|\varphi_0\rangle\langle\varphi_0|\varphi_1\rangle\langle\varphi_1|)$ is independent on the value of d. Then we can get the bipartite nonlocal information for partition $1|234$ as $S'_{1|234} = 2 - 2(p_0^2 d^4 + p_1^2/d^4 + f_1)/Q^4$. In the same way, after straightforward calculation, we can get the other bipartite nonlocal information $S'_{i|\tilde{i}} = 2 - 2[p_0^2 d^4 Tr\rho_i^2(0) + p_1^2/d^4 Tr\rho_i^2(1) + f_i]/Q^4$ and $S'_{ij|\tilde{ij}} = 2 - 2[p_0^2 d^4 Tr\rho_{ij}^2(0) + p_1^2/d^4 Tr\rho_{ij}^2(1) + f_{ij}]/Q^4$, where $\tilde{i}$, $\tilde{ij}$ represent the qubits other than i and i, j, $\rho_i(k)$ $(i = 2, 3, 4)$ are the reduce density matrices of the ith qubit and $\rho_{ij}(k)$ $(ij = 23, 24, 34)$

are the reduce density matrix of the ith and jth qubit from $|\varphi_k\rangle$ $(k=0,1)$. f_i and f_{ij} are all independent on the value of d. We note that $Tr\rho_2^2(k)=Tr\rho_{34}^2(k)$, $Tr\rho_3^2(k)=Tr\rho_{24}^2(k)$ and $Tr\rho_4^2(k)=Tr\rho_{23}^2(k)$. It can been seen that $\mathcal{E}_{1234}(|\psi'\rangle\langle\psi'|)=2(2p_0p_1-f_1-f_2-f_3-f_4+f_{23}+f_{24}+f_{34})/Q^4$, i.e.

$$\mathcal{E}_{1234}(|\psi'\rangle\langle\psi'|)=\mathcal{E}_{1234}(|\psi\rangle\langle\psi|)/Q^4 \tag{3}$$

If $|\psi\rangle$ is a product pure state, $\mathcal{E}_{1234}$ is always zero, i.e. it satisfies the above Eq.(3) too. Therefore, if we take into account the normalization factor, $\mathcal{E}_{1234}$ is SLOCC invariant [33].

We now prove that $\mathcal{E}_{1234}$ satisfies the above condition (2), i.e. $\mathcal{E}_{1234}\geq 0$. It has been shown that [33] all pure multipartite states can be transformed into a normal form, all local density operators of which are proportional to the identity, by the determinant 1 SLOCC operations. A generic pure state of four qubits can always be transformed to the normal form state [34] $G_{abcd}=\frac{a+d}{2}(|0000\rangle+|1111\rangle)+\frac{a-d}{2}(|0011\rangle+|1100\rangle)+\frac{b+c}{2}(|0101\rangle+|1010\rangle)+\frac{b-c}{2}(|0110\rangle+|1001\rangle)$, where a,b,c,d are complex parameters with non-negative real part. We denote $A=(a+d)/2, B=(b+c)/2, C=(a-d)/2, D=(b-c)/2$. If G_{abcd} is a product state, $\mathcal{E}_{1234}=0$. Again we assume G_{abcd} is a non-product state in the following. The bipartite nonlocal information of different partitions is give by $S_{1|234}=S_{2|134}=S_{3|124}=S_{4|123}=1$. $S_{12|34}$, $S_{13|24}$ and $S_{14|23}$ can be obtained from $Tr\rho_{12}^2=\mathcal{T}(A,B,C,D)$, $Tr\rho_{13}^2=\mathcal{T}(A,C,B,D)$ and $Tr\rho_{14}^2=\mathcal{T}(A,B,D,C)$, where $\mathcal{T}(x_1,x_2,x_3,x_4)=2\{(|x_1|^2+|x_3|^2)^2+(|x_2|^2+|x_4|^2)^2+|x_1x_3^*+x_1^*x_3|^2+|x_2x_4^*+x_2^*x_4|^2\}/\mathcal{M}^2$ with $\mathcal{M}=2(|x_1|^2+|x_2|^2+|x_3|^2+|x_4|^2)$. We first prove the following inequality $\sum_{i=1}^4|x_i|^4+\sum_{\substack{i=1\\j>i}}^4|x_ix_j^*+x_i^*x_j|^2\geq 2\sum_{\substack{i=1\\j>i}}^4|x_i|^2|x_j|^2$. Without loss of generality, we could assume that $x_1=|x_1|\in\mathcal{R}$ and $x_i=|x_i|\exp{(i\phi_i/2)}$ with $i=2,3,4$. The inequality is equivalent to $\sum_{i=1}^4|x_i|^4+2(\sum_{i=2}^4|x_1|^2|x_i|^2\cos\phi_i+|x_2|^2|x_3|^2\cos{(\phi_2-\phi_3)}+|x_2|^2|x_4|^2\cos{(\phi_2-\phi_4)}+|x_3|^2|x_4|^2\cos{(\phi_3-\phi_4)})=(|x_1|^2+\sum_{i=2}^4|x_i|^2\cos\ \phi_i)^2+(\sum_{i=2}^4|x_i|^2\sin\ \phi_i)^2\geq 0$. According to this inequality, is can be easily verified that

$$\mathcal{E}_{1234}(G_{abcd})\geq 0 \tag{4}$$

Together with Eq.(3), we obtain that $\mathcal{E}_{1234}(|\psi\rangle\langle\psi|)\geq 0$ for any four-qubit pure state.

The remain condition is that $\mathcal{E}_{1234}$ should be an entanglement monotone. Note that any local protocol can be decomposed into local positive-operator valued measure (POVMs) that can be implemented by a sequence of two-outcome POVMs performed by one party on the system [32]. Without loss of generality, we only need to consider the general two-outcome POVMs performed on the $1st$ qubit $\{M_1,M_2\}$, such that $M_1^\dagger M_1+M_2^\dagger M_2=\mathcal{I}$. Using the singular value decomposition, we can write $M_1=U_1\cdot diag\{a,b\}\cdot V, M_2=U_2\cdot diag\{\sqrt{1-a^2},\sqrt{1-b^2}\}\cdot V$, where U_1, U_2 and V are unitary matrices. We denote that $M_1'=M_1/(ab)^{1/2}, M_2'=M_2/\{(1-a^2)(1-b^2)\}^{1/4}$. Note that $\det M_1'=\det M_2'=1$. According to Eq.(3), we get $\mathcal{E}_{1234}(|\psi_1\rangle\langle\psi_1|)=\frac{a^2b^2}{Q_1^4}\mathcal{E}_{1234}(|\psi\rangle\langle\psi|)$, $\mathcal{E}_{1234}(|\psi_2\rangle\langle\psi_2|)=\frac{(1-a^2)(1-b^2)}{Q_2^4}\mathcal{E}_{1234}(|\psi\rangle\langle\psi|)$, where $|\psi_i\rangle=M_i|\psi\rangle/Q_i$ and Q_i are normalization factors. After simple algebra calculation and using the fact that the arithmetic mean always exceeds the geometric mean [8, 32, 33], we can see that

$$Q_1^2\mathcal{E}_{1234}(|\psi_1\rangle\langle\psi_1|)+Q_2^2\mathcal{E}_{1234}(|\psi_2\rangle\langle\psi_2|)\leq\mathcal{E}_{1234}(|\psi\rangle\langle\psi|)$$

is fulfilled, i.e. $\mathcal{E}_{1234}$ is an entanglement monotone. Therefore, it satisfies all the necessary conditions for a natural entanglement measure.

General even number qubits The above discussions can be extended to the situation when N is a generic even number. We denote the set of all genuine $k-$qubit entanglement that contributes to the bipartite nonlocal information of partition $\mathcal{P}\in\mathcal{P}_{\mathcal{I}}$ and $\mathcal{P}\in\mathcal{P}_{\mathcal{II}}$ as $\mathcal{EC}_{\mathcal{I}}^k$, $\mathcal{EC}_{\mathcal{II}}^k$ respectively. If $k=N$, the contribution is just genuine $N-$qubit nonlocal information $I_{12\cdots N}$, and $|\mathcal{EC}_{\mathcal{I}}^N|-|\mathcal{EC}_{\mathcal{II}}^N|=1$. This can be seen from the polynomial expansion of $(1-x)^N|_{x=1}=(2|\mathcal{EC}_{\mathcal{II}}^N|+2)-2|\mathcal{EC}_{\mathcal{I}}^N|=0$. For $2\leq k<N$, if the nonlocal information $I_{a_1a_2\cdots a_lb_1b_2\cdots b_{k-l}}$ $(a_1,a_2,\cdots,a_l\in\mathcal{A},\ b_1,b_2,\cdots,b_{k-l}\in\mathcal{B})$ contributes to the bipartite nonlocal information of some partition $\mathcal{P}=\mathcal{A}|\mathcal{B}\in\mathcal{P}_{\mathcal{I}}$, there must exist one maximum index, denoted as $\mathcal{X}$, which does not belong to $\{a_1,a_2,\cdots,a_l,b_1,b_2,\cdots,b_{k-l}\}$. If $\mathcal{X}\in\mathcal{A}$ or $\mathcal{B}$, then we can construct a bipartite partition $\mathcal{P}'=\mathcal{A}-\{\mathcal{X}\}|\mathcal{B}+\{\mathcal{X}\}$ or $\mathcal{A}+\{\mathcal{X}\}|\mathcal{B}-\{\mathcal{X}\}\in\mathcal{P}_{\mathcal{II}}$. The nonlocal information $I_{a_1a_2\cdots a_lb_1b_2\cdots b_{k-l}}$ will also contributes to the bipartite nonlocal information of partition $\mathcal{P}'$. According to this one to one homologous relation, we can obtain that $\sum_{k=2}^N(\mathcal{EC}_{\mathcal{I}}^k-\mathcal{EC}_{\mathcal{II}}^k)=I_{12\cdots N}$. Therefore, a measure for genuine $N-$qubit entanglement, $N\in even$, of pure states can be defined naturally as follows

$$\mathcal{E}_{12\cdots N}=\sum_{\mathcal{P}\in\mathcal{P}_{\mathcal{I}}}S_{\mathcal{P}}-\sum_{\mathcal{P}\in\mathcal{P}_{\mathcal{II}}}S_{\mathcal{P}} \tag{5}$$

where $S_{\mathcal{P}}$ is the bipartite nonlocal information for partition $\mathcal{P}$.

$\mathcal{E}_{12\cdots N}$ is SLOCC invariant, and is unchanged under permutations of qubits, i.e., it represents a collective property of all the N qubits. Our measure can surely distinguish two different kinds of multi-qubit entangled states, general $N-$qubit GHZ states $|GHZ\rangle_N=\frac{1}{\sqrt{2}}(|00\cdots 0\rangle+|11\cdots 1\rangle)$ and W states $|W\rangle_N=\frac{1}{\sqrt{N}}(|00\cdots 01\rangle+|00\cdots 010\rangle+\cdots+|10\cdots 0\rangle)$, for $\mathcal{E}_{12\cdots N}(|GHZ\rangle_N)=1$ and $\mathcal{E}_{12\cdots N}(|W\rangle_N)=0$. Although we have not proved that $\mathcal{E}_{12\cdots N} \geq 0$ for $N > 4$, which is actually related to the intricate compatibility problem of multipartite pure states [35], we have calculated numerically $\mathcal{E}_{12\cdots N}$ for more than 10^5 arbitrarily chosen pure states of six and eight qubits. The numerical results suggest strongly that $\mathcal{E}_{12\cdots N} \geq 0$. The calculation of $\mathcal{E}_{12\cdots N}$ is very straightforward. It can indeed be determined from observable quantities [10, 27], which can be conveniently measured in experiments.

General odd number qubits The above results are not applicable to the situation of odd number qubits straightforwardly. However, it is easy to verify that the genuine $N-$qubit entanglement, $N \in odd$, can be characterized by the bipartite nonlocal information of different partitions together with genuine $(N-1)-$qubit entanglement, where $N-1$ is an even number. Therefore, the measure can also be obtained based on our idea in principle.

Based on the measure for pure states of N qubits, the measure for $N-$qubit mixed states is defined by the convex roof extension of pure-state measure according to the standard entanglement theory [16], i.e.

$$\mathcal{E}_{12\cdots N}(\rho) = \min \sum_k p_k \mathcal{E}_{12\cdots N}(|\psi_k\rangle\langle\psi_k|) \tag{6}$$

where min runs through all possible decompositions of ρ into pure states, *i.e.*, $\rho = \sum\limits_k p_k |\psi_k\rangle\langle\psi_k|$.

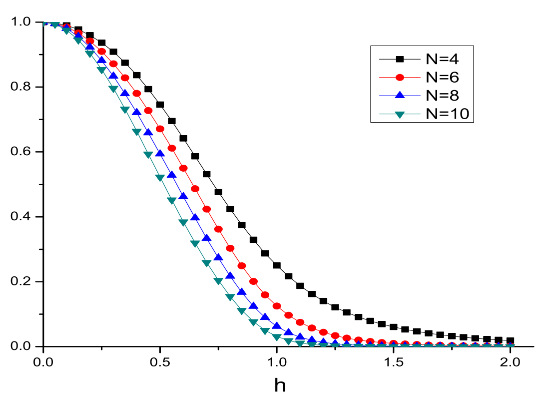

FIG. 2: (Color online) Genuine $N-$qubit entanglement $\mathcal{E}_{12\cdots N}$ for the ground state of the finite transverse field Ising model with $N = 4, 6, 8, 10$.

Genuine multi-qubit entanglement in spin systems Our measure for genuine multi-qubit entanglement will extend the research of the relation between entanglement and quantum phase transitions. Given a quantum system with N spins, one can compute the genuine N-qubit entanglement of the ground state for different even number N. In addition, the translational invariant property together with other symmetries of the system Hamiltonian will simplify the calculation of $\mathcal{E}_{12\cdots N}$ significantly. As an illustration, we consider the ground state of the transverse field Ising Hamiltonian $\mathcal{H} = -\sum\limits_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x - h \sum\limits_{i=1}^{N} \sigma_i^z$. We plot the behavior of $\mathcal{E}_{12\cdots N}$ for different system size $N = 4, 6, 8, 10$ (see Fig. 2). Using the standard finite-size scaling theory, it can be seen that at the quantum critical point $h_c = 1$, genuine $N-$qubit entanglement changes drastically. Compared to the results in Refs. [23], the behavior of $\mathcal{E}_{12\cdots N}$ is very similar to LE, i.e. it can capture the feature of long-range quantum correlations.

In conclusion, we have introduced an information-theoretic measure of genuine multi-qubit entanglement, which is the collective property of the whole state. For pure states of an even number of qubits, this measure is easily computable, and are dependent on observable quantities. Therefore the measurement in experiments is convenient. For the pure states of an odd number of qubits, the measure is defined based on the results for the states of an even number qubits. Finally, we demonstrated the usefulness of our measure in spin systems. Further generalizations and application will be presented in our future work. Our results will help gain important insight into the structure and nature of multipartite entanglement, and enlighten the research of quantum entanglement in condensed matter physics.

Acknowledgment We gratefully acknowledge valuable discussions with Lu-Ming Duan, Yong Hu, Yong-Jian Han and Xiang-Fa Zhou. This work was funded by National Fundamental Research Program (2001CB309300), the Innovation funds from Chinese Academy of Sciences, NCET-04-0587, and National Natural Science Foundation of China (Grant No. 60121503, 10574126).

[1] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47, 777 (1935).
[2] R.Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 005188 (2001).
[3] Zhi Zhao, Yu-Ao Chen, An-Ning Zhang, Tao Yang, Hans Briegel, Jian-Wei Pan, Nature (London) 430, 54 (2004).
[4] Č. Brukner, V. Vedral, A. Zeilinger, quant-ph/0410138; T. Roscilde, P. Verrucchi, A. Fubini, S. Haas, and V. Tognetti, Phys. Rev. Lett. 93, 167203 (2004) ; Phys. Rev. Lett. 94, 147208 (2005).
[5] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) 416, 608 (2002); G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
[6] T. J. Osborne, and M. A. Nielsen, Phys. Rev. A 66, 032110 (2002).
[7] V. Vedral, New J. Phys. 6, 22 (2004).
[8] A. Wong and N. Christensen, Phys. Rev. A 63, 044301 (2001).
[9] A. Miyake, Phys. Rev. A 67, 012108 (2003).
[10] Peter J. Love, Alec Maassen van den Brink, A. Yu. Smirnov, M. H. S. Amin, M. Grajcar, E. Il'ichev, A. Izmalkov, A. M. Zagoskin, quant-ph/0602143.

[11] D. A. Meyer and N. R. Wallach, J. Math. Phys. 43, 4273 (2002); A. J. Scott, Phys. Rev. A 69, 052330 (2004); G. K. Brennen, Quant. Inf. and Comput. 3, 616 (2003).
[12] Clive Emary, quant-ph/0405049; Gerardo A. Paz-Silva, John H. Reina, quant-ph/0603102; P. Facchi, G. Florio and S. Pascazio, quant-ph/0603281.
[13] Florian Mintert, Marek Kus, Andreas Buchleitner, Phys. Rev. Lett. 95, 260502 (2005); Florian Mintert, Andreas Buchleitner, quant-ph/0411130.
[14] Howard Barnum, Emanuel Knill, Gerardo Ortiz, Rolando Somma, and Lorenza Viola, Phys. Rev. Lett. 92, 107902 (2004).
[15] D.L. Zhou, B. Zeng, Z. Xu, L. You, quant-ph/0608240.
[16] M. B. Plenio, S. Virmani, quant-ph/0504163 and reference therein.
[17] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A 54, 3824 (1996).
[18] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[19] A. Peres, Phys. Rev. Lett. 76, 1413 (1996); K. Zyczkowski and P. Horodecki, Phys. Rev. A 58, 883 (1998).
[20] V. Vedral, M. B. Plenio, M.A. Rippin, and P.L. Knight, Phys. Rev. Lett. 78, 2275 (1997).
[21] D. P. DiVincenzo, C. A. Fuchs, H. Mabuchi, J. A. Smolin, A. Thapliyal, A. Uhlmann, quant-ph/9803033; O. Cohen, Phys. Rev. Lett. 80, 2493 (1998).
[22] Florian Mintert, Marek Kus, Andreas Buchleitner, Phys. Rev. Lett. 92, 167902 (2004).
[23] F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. 92, 027901 (2004).
[24] C. Lunkes, Č Brukner, and V. Vedral, Phys. Rev. Lett. 95, 030503 (2005) .
[25] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A 63, 012307 (2001).
[26] M. S. Leifer and N. Linden, A. Winter, Phys. Rev. A 69, 052304 (2004).
[27] Leandro Aolita and Florian Mintert, Phys. Rev. Lett. 97, 50501 (2006); S. P. Walborn, P. H. Souto, L. Davidovich, F. Mintert, and A. Buchleitner, Nature (London) 440, 1022 (2006).
[28] M. Horodecki and R. Horodecki, Phys.Lett. A 244, 473 (1998); M. Horodecki, K. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), and U. Sen, Phys. Rev. Lett. 90, 100402 (2003); M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak-Radtke, Phys. Rev. A 71, 062307 (2005).
[29] J.M. Cai, Z.W. Zhou and G.C. Guo, quant-ph/0609026.
[30] E. Santos and M. Ferrero, Phys. Rev. A 62, 024101 (2000).
[31] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
[32] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[33] F. Verstraete, J. Dehaene, and B. De Moor, Phys. Rev. A 68, 012103 (2003).
[34] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A 65, 052112 (2002).
[35] N. Linden, S. Popescu, and W. K. Wootters, Phys. Rev. Lett. 89, 207901 (2002); N. Linden and W. K. Wootters, Phys. Rev. Lett. 89, 277906 (2002); A. Higuchi, A. Sudbery, and J. Szulc, Phys. Rev. Lett. 90, 107902 (2003).